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Non-Abelian Born–Infeld and kappa-symmetry

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We define an iterative procedure to obtain a non-Abelian generalization of the Born–Infeld action. This construction is made possible by the use of the severe restrictions imposed by kappa-symmetry. In this paper we will present all bosonic terms in the action up to terms quartic in the Yang–Mills field strength and all fermion bilinear terms up to terms cubic in the field strength. Already at this order the fermionic terms do not satisfy the symmetric trace-prescription. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1374449]

I. INTRODUCTION

One of the most intriguing features of D-branes is their close connection with gauge theories. Indeed, the effective theory describing the worldvolume dynamics of a Dp-brane is a $p+1$-dimensional field theory with, in the static gauge, as bosonic degrees of freedom the transversal coordinates of the brane, appearing as $9-p$ scalar fields, and the massless states of the open strings ending on the brane which appear as a U(1) gauge field. When these fields vary slowly, the effective action governing their dynamics is known to all orders in α'. It is the ten-dimensional Born–Infeld action,[1] dimensionally reduced to $p+1$ dimensions.

Once several D-branes are present, the situation changes. The mass of the strings stretching between two branes is proportional to the shortest distance between the branes. Starting off with n well separated D-branes we end up with a $\mathrm{U}(1)^n$ theory, however, once the n branes coincide additional massless states appear which complete the gauge multiplet to a non-Abelian U(n)-theory.[2] Contrary to the Abelian case, the effective action is not known to all orders in α'. The first term, quadratic in the field strength, is nothing but a dimensionally reduced U(n) Yang–Mills theory. The next order, which is quartic in the field strength, was obtained from the four-gluon scattering amplitude in open superstring theory[3] and from a three-loop β-function calculation.[4] Based upon these results and other considerations, an all order proposal was formulated for the effective action;[5] the non-Abelian Born–Infeld action assumes essentially the same form as the Abelian one, however, all Lie algebra valued objects have to be symmetrized first before taking the trace. Other trace prescriptions, involving commutators, have been given as well.[6] More recently, it was found that the symmetric trace prescription could not be correct as it did not reproduce the mass spectrum of certain D-brane configurations.[7,8] It was shown in Ref. 9 that by adding commutator terms to the action the problem might be cured. Indeed, as was pointed out in Ref. 1, the notion of an effective action for slowly varying fields is subtle in the non-Abelian case. In the effective action higher derivative terms are dropped. However because of

$$D_iD_jF_{kl}=\frac{1}{2}\{D_i,D_j\}F_{kl}-\frac{i}{2}[F_{ij},F_{kl}], \tag{1.1}$$

this is ambiguous. The analysis of the mass spectrum seems to indicate that the symmetrized product of derivatives acting on a field strength should be viewed as an acceleration term which can safely be neglected, while the anti-symmetrized products should be kept. A systematic study of the F^6 terms[10] showed that using the mass spectrum as a guideline, almost all terms at this order could be determined. However due to the specific choices of backgrounds made in Ref. 10, certain terms do not contribute to the mass spectrum and as a result can not be fixed in this way. A direct calculation from a six-point open superstring amplitude or a five-loop β-function seems unfeasible, so another approach is called for.

Until now, we ignored the fermionic degrees of freedom in our discussion. The fully covariant worldvolume theory of a single D-brane in a type II theory can be formulated in terms of the following world volume fields: the embedding coordinates $X^{\mu}(\sigma)$ (of which only the transverse coordinates represent physical degrees of freedom), the Born–Infeld vector field $V_i(\sigma)$, and $N=2$ space–time fermionic fields $\theta(\sigma)$. In a curved background the D-brane can be coupled to the corresponding type II supergravity superfields, and $N=2$ supersymmetry is realized locally. In a flat background there is $N=2$ global supersymmetry. This world volume theory has a local κ-symmetry, which acts on the fermions as

$$\delta\overline{\theta}(\sigma)=\overline{\kappa}(\sigma)(1+\Gamma), \tag{1.2}$$

where Γ, which depends on world volume as well as background fields, satisfies

$$\Gamma^2=\mathbb{1}. \tag{1.3}$$

The projection (1.2) makes it possible to gauge away half the fermionic degrees of freedom. The field content then corresponds in a static gauge to that of a supersymmetric Yang–Mills theory in $p+1$ dimensions. There is still $N=2$ supersymmetry, but half of this is realized nonlinearly. These covariant D-brane actions have been constructed in flat,[11] as well as in curved backgrounds.[12,13] This paper examines the suggestion in Ref. 8 that κ-symmetry might teach us something about the orderings appearing in the non-Abelian Born–Infeld action.

D-brane actions consist of the sum of a Born–Infeld term, coupling the world volume fields to the NS–NS sector of the background, and a Wess–Zumino term in which the couplings to the R–R fields occur. Each part is separately supersymmetric, but the two are related, in the Abelian case, by the κ-transformations. The Wess–Zumino term is of a topological nature, and can therefore be formulated in a metric-independent way. Its structure is severely restricted, also in the non-Abelian case. The Born–Infeld term is much more complicated, and consequently its generalization from the Abelian to the non-Abelian case is much more difficult. It is natural to assume that also in the non-Abelian case the Born–Infeld and Wess–Zumino term are related by a κ-symmetry. Our aim is to use the knowledge of the Wess–Zumino term and the properties of κ-symmetry to obtain information about the non-Abelian Born–Infeld term.

To construct this non-Abelian generalization we will use an iteration in the number of Yang–Mills field strengths $F(V)$. In this paper we will obtain all terms in the action up to and including the order F^2. As we discussed above, in the purely bosonic terms the conflict between string theoretic results and the symmetric trace prescription arises only at order F^6, so it is clear that in this paper we will not contribute to the discussion of these bosonic terms. However, we find that in the fermionic sector already at quadratic order some of the fermionic terms do not correspond to a symmetric trace.

This paper is organized as follows: We will discuss our choice of variables in Sec. II. In Sec. III we will define the iterative procedure and illustrate it for the Abelian case. In Secs. IV and V we derive and present our results for the non-Abelian case, and bring them to gauge fixed form in Sec. VI. In Sec. VII we give our conclusions, and point out a number of extensions and applications of this work.

We will end the Introduction by recalling briefly some related work on supersymmetric D-brane actions. In four dimensions the supersymmetrization of the Abelian Born–Infeld action in $N=1$ supersymmetry has been known for a long time.[14] More recently, this work has been extended to the non-Abelian Born–Infeld theory, and to $N=2$ supersymmetry.[15,16] In particular, in Ref. 15 it is remarked that $N=2$ supersymmetry in four dimensions is not sufficient to resolve the ordering ambiguities, several ordering prescriptions give rise to supersymmetric actions. So it seems that it is indeed necessary (and hopefully sufficient) to consider $N=4$ in $D=4$, or, as in this paper, the ten-dimensional supersymmetric Born–Infeld action action. Several aspects of the ten-dimensional problem have been studied in Refs. 17 and 18. In particular, these authors investigate the dependence of the action on transverse scalars, where these scalars are generated by T-duality starting from the D9-brane action. However, this is in the context of the symmetric trace prescription.

II. WORLD VOLUME FIELDS AND TRANSFORMATIONS

The aim of this work is to obtain the effective action for n overlapping Dp-branes, with U(n) covariance on the world volume. Before embarking on the construction, one has to carefully choose the starting point of the calculation. For general Dp-branes the situation is complicated by the presence of the transverse scalar degrees of freedom, which are in the adjoint representation of the Yang–Mills group. Not only does one have to take commutators of these scalars into account, but also the background fields will depend on these scalars.[19] We avoid these complications by limiting ourselves to the case of n overlapping D9-branes, and by choosing a flat background. Through T-duality D-branes for other values of p can be obtained, the extension to curved backgrounds will be discussed in Sec. VII.

For n overlapping D9-branes the completely gauge fixed result should be the supersymmetric version of the non-Abelian Born–Infeld theory. Since the vector fields $V_i^A(\sigma)$, $A=1,...,n^2$, are in the adjoint representation of U(n), we have to make the same choice for the fermion fields θ. Therefore we start out with fields $\theta^A(\sigma)$, which form a doublet $(N=2)$ of Majorana–Weyl spinors for each A, satisfying $\Gamma_{11}\theta^A=\theta^A$. After κ-gauge fixing only half of each doublet will remain, and we have the correct number of degrees of freedom for the supersymmetric Yang–Mills theory.

This requires, that there are as many κ-symmetries as θ's, so that the parameter of the κ transformations will have to be in the adjoint of U(n) as well. Thus the θ^A transform as follows under ordinary supersymmetry (ϵ), κ-symmetry (κ), Yang–Mills transformations (Λ^A), and world volume reparametrizations (ξ^i),

$$\delta\bar{\theta}^A(\sigma)=-\bar{\epsilon}^A+\bar{\kappa}^B(\sigma)(\mathbb{1}\delta^{BA}+\Gamma^{BA}(\sigma))+f^A{}_{BC}\Lambda^B(\sigma)\bar{\theta}^C(\sigma)+\xi^i(\sigma)\partial_i\bar{\theta}^A(\sigma). \tag{2.1}$$

Here ϵ^A are constant, Γ^{AB} depends on the world volume fields, and therefore on σ. It must satisfy

$$\Gamma^{AB}\Gamma^{BC}=\delta^{AC}\mathbb{1}. \tag{2.2}$$

We will usually write these transformations in terms of

$$\delta\bar{\theta}^A\equiv\bar{\eta}^A\equiv\bar{\kappa}^B(\sigma)(\mathbb{1}\delta^{BA}+\Gamma^{BA}(\sigma)). \tag{2.3}$$

Useful information is obtained by considering commutators of these transformations. Because ϵ^A is constant we find from the commutator of Yang–Mills and supersymmetry transformations that

$$f^A{}_{BC}\Lambda^B\epsilon^C=0\rightarrow f_{ABC}\epsilon^C=0. \tag{2.4}$$

Therefore $\epsilon=\epsilon^A T_A$, where T_A are the U(n) generators, must be proportional to the unit matrix, i.e., we can choose a basis in which there is only one nonvanishing ϵ parameter. So only a subset of the θ^A transform under supersymmetry, and there is only one independent supersymmetry parameter. The θ's which are presently inert under supersymmetry will obtain their supersymmetry transformations through κ-gauge fixing, as we shall see in Sec. VI. From the commutator of κ-symmetry and supersymmetry we find $\delta_\epsilon\eta^A=0$. This implies that

$$\delta_\epsilon\Gamma^{AB}=0. \tag{2.5}$$

The only scalars we have are the embedding coordinates $X^{\mu}(\sigma)$ for world volume directions. There are several options that one could consider for the X^{μ}:

(1) We could assume that we are in the static gauge, i.e.,
$$X^{\mu}(\sigma)=\delta_i^{\mu}\sigma^i, \tag{2.6}$$
from the beginning, so that the X^{μ} are absent. In this case there are no world volume reparametrizations, i.e., $\xi^i=0$ in (2.1);

(2) We could decide that the X^{μ} are in the singlet representation of the Yang–Mills group. The idea is that the n branes overlap, there is only one set of world volume coordinates, and the corresponding reparametrization group would be sufficient to gauge fix a singlet set of embedding coordinates;

(3) We could choose the X^{μ} in the adjoint representation of Yang–Mills in analogy with transverse coordinates for $p<9$. Here one thinks of starting with n separate branes where each has its own world volume and embedding coordinates. When the branes overlap the embedding coordinates ''fill up'' to form elements of the adjoint representation. Clearly this requires a different approach toward the worldvolume reparametrisation invariance, which must then correspond to a sufficiently large symmetry group to gauge fix all these embedding functions.

We have investigated the first two possibilities in the non-Abelian case, and we have found that only the first approach is consistent with the iterative procedure that we employ. In Sec. IV we will point out where the first two choices start to diverge, in Sec. VII we will briefly come back to the third possibility.

The transformation rules of the bosonic fields V_i^A, and, in case of the second choice above, of the X^{μ}, are determined iteratively by requiring invariance of the action.

A special case of a U(n) invariant non-Abelian D-brane action is of course the truncation to $\mathrm{U}(1)^n$. In this case we know the answer: a κ-invariant action is given by the sum of n Abelian D-brane actions. This special case will be discussed again, since it plays a role in making a choice between the different possibilities for the variables X^{μ} discussed above.

Throughout this paper we will limit ourselves to terms in the action and transformation rules which are at most quadratic in the fermion fields.

III. THE ITERATIVE PROCEDURE AND THE ABELIAN EXAMPLE

In this section we will set up our iterative procedure and illustrate it for the Abelian case. To do this, we must first give some details of the effective D9-brane action in a flat background.[11] In this case we can use a covariant formulation with embedding coordinates X^{μ}, space–time fermions θ, and the Born–Infeld vector V_i. They transform under supersymmetry (ϵ), κ-symmetry, world volume reparametrizations (ξ^i), and Maxwell gauge transformations (Λ) as

$$\delta X^{\mu}=\tfrac{1}{2}\bar{\epsilon}\Gamma^{\mu}\theta+\tfrac{1}{2}\bar{\eta}\Gamma^{\mu}\theta+\xi^i\partial X^{\mu},$$

$$\delta\bar{\theta}=-\bar{\epsilon}+\bar{\eta}+\xi^i\partial_i\bar{\theta},$$

$$\delta V_i=-\tfrac{1}{2}(\bar{\epsilon}+\bar{\eta})\gamma_i\theta+\xi^jF_{ji}+\partial_i(\Lambda+\xi^jV_j), \tag{3.1}$$

with

$$\bar{\eta}=\bar{\kappa}(1+\Gamma),\quad (\Gamma)^2=\mathbb{1}, \tag{3.2}$$

and

$$\gamma_i\equiv\Gamma^{\mu}\partial_iX^{\mu}. \tag{3.3}$$

The Born–Infeld contribution reads

$$\begin{aligned}\mathcal{L}_{BI} &= -\sqrt{-\det(g+\mathcal{F})}\\ &= -\sqrt{-\det g}\left(1+\tfrac{1}{4}\mathcal{F}_{ij}\mathcal{F}^{ij}+\cdots\right),\end{aligned} \tag{3.4}$$

where in the second line we expand to second order in $\mathcal{F}$, which is given by

$$\mathcal{F}_{ij}=F_{ij}(V)-B_{ij}\,. \tag{3.5}$$

In a flat background and in the quadratic fermion approximation, B_{ij} is

$$B_{ij}=-\bar{\theta}\sigma_3\gamma_{[i}\partial_{j]}\theta. \tag{3.6}$$

The world volume metric reads

$$g_{ij}=\eta_{ij}+\bar{\theta}\gamma_{(i}\partial_{j)}\theta, \qquad \eta_{ij}\equiv\partial_i X^\mu\partial_j X_\mu\,. \tag{3.7}$$

The metric g and $\mathcal{F}$ are invariant under supersymmetry and transform covariantly under κ-transformations. The most useful form for comparison with the non-Abelian case is the expansion of (3.4) to second order in fermions,

$$\mathcal{L}_{BI}=-\sqrt{-\det\eta}\left(1+\tfrac{1}{2}\bar{\theta}\gamma^i\partial_i\theta+\tfrac{1}{2}\bar{\theta}\sigma_3\gamma_{[i}\partial_{j]}\theta F^{ij}+\tfrac{1}{4}F^{ij}F_{ij}+\tfrac{1}{2}\bar{\theta}\gamma_i\partial_j\theta T^{ij}+\cdots\right), \tag{3.8}$$

where T^{ij} is the energymomentum tensor of the vector field,

$$T^{ij}=F^{ik}F_k^{\ j}+\tfrac{1}{4}\eta^{ij}F_{kl}F^{kl}. \tag{3.9}$$

The Wess–Zumino term takes on the following form:

$$\mathcal{L}_{WZ}=e^{i_1\cdots i_{10}}\times\sum_{k=0}^{4}\frac{(-1)^k}{2^{k+1}k!(9-2k)!}\bar{\theta}\mathcal{P}_{(k)}\gamma_{i_1\cdots i_{9-2k}}\partial_{i_{10-2k}}\theta(\mathcal{F}^k)_{i_{11-2k}\cdots i_{10}}, \tag{3.10}$$

where

$$\mathcal{P}_{(k)}=\sigma_1\ \ (\text{for}\ \ k=0,2,4),\ \ \mathcal{P}_{(k)}=i\sigma_2(\text{for}\ \ k=1,3). \tag{3.11}$$

Note that the sum in (3.10) runs only to $k=4$, since the RR-scalar field vanishes in the flat background.

It will be useful, also for the non-Abelian case, to discuss why we make this particular choice for the $\mathcal{P}_{(k)}$. The $\mathcal{P}_{(k)}$ are chosen such that the contributions to the Wess–Zumino term are not total derivatives. For odd k this fixes $\mathcal{P}_{(k)}$ to be $i\sigma_2$. For even k we could also have chosen $\mathbb{1}$ or σ_3. When we start looking at the iterative procedure later in this Section, we will find that we need

$$\{\mathcal{P}_{(0)},\mathcal{P}_{(1)}\}=0, \tag{3.12}$$

which excludes $\mathbb{1}$ for $k=0$. We have in principle the possibility to have either σ_1 or σ_3 (or both) for $k=0$, and taking σ_1 is a choice of basis for the $N=2$ fermions. Note that σ_3 in (3.6) is correlated with the choice for σ_1 in the Wess–Zumino term: had we chosen $\mathcal{P}_{(0)}=\sigma_3$ we would have found σ_1 in (3.6).

The structure of (3.10) guarantees that the WZ-action transforms into a total derivative under the global supersymmetry transformation $\delta\bar{\theta}=-\bar{\epsilon}$. Since we do not go beyond bilinear fermions we can use F instead of $\mathcal{F}$ in the Wess–Zumino term. The numerical coefficients in (3.10) are determined by κ-symmetry once the normalization of F and θ are fixed in the Born–Infeld term.

Let us now consider the iterative construction of κ-symmetry. The variation of the D-brane action takes on the form,

$$\delta\mathcal{L} = -\overline{\eta}(1-\Gamma)\mathcal{T}. \tag{3.13}$$

It indeed vanishes if η is given by (3.2). These variations have the following source:

$$\delta\mathcal{L}_{BI} = -\overline{\eta}\mathcal{T}, \quad \delta\mathcal{L}_{WZ} = \overline{\eta}\Gamma\mathcal{T}. \tag{3.14}$$

The variation of the Wess–Zumino term, together with the information that $\Gamma^2 = \mathbb{1}$, is sufficient to determine both Γ and $\mathcal{T}$ iteratively. Since $\mathcal{T}$ determines the variation of the Born–Infeld term this information is sufficient to obtain iteratively the Born–Infeld part of the action.

The iteration is obtained by expanding Γ and $\mathcal{T}$ in F,

$$\begin{aligned}\delta\mathcal{L} &= -\overline{\eta}(1-(\Gamma_0+\Gamma_1+\cdots))(\mathcal{T}_0+\mathcal{T}_1+\cdots)\\ &= -\overline{\eta}(\mathcal{T}_0-\Gamma_0\mathcal{T}_0+\mathcal{T}_1-\Gamma_1\mathcal{T}_0-\Gamma_0\mathcal{T}_1+\cdots),\end{aligned} \tag{3.15}$$

where the indices indicate the order in F and the Γ_i satisfy various identities which follow from $\Gamma^2 = \mathbb{1}$. Since it will be useful to have the Abelian results at hand for comparison with the non-Abelian calculation in Sec. IV, we will work out the beginning of this iteration in some detail.

Let us start with the order F^0. The contribution from (3.10) is

$$\mathcal{L}_{WZ0} = \frac{1}{2\cdot 9!}\,\epsilon^{i_1\cdots i_{10}}\overline{\theta}\sigma_1\gamma_{i_1\cdots i_9}\partial_{i_{10}}\theta. \tag{3.16}$$

For the κ-variation we only have to vary θ to obtain

$$\delta\mathcal{L}_{WZ0} = \frac{1}{9!}\,\epsilon^{i_1\cdots i_{10}}\overline{\eta}\sigma_1\gamma_{i_1\cdots i_9}\delta_{i_{10}}\theta = \sqrt{-\det\eta}\,\overline{\eta}\sigma_1\Gamma^{(0)}\gamma^i\partial_i\theta. \tag{3.17}$$

Here $\Gamma^{(0)}$ is given by

$$\Gamma^{(0)} = \frac{1}{10!\sqrt{-\det g}}\,\epsilon^{i_1\cdots i_{10}}\gamma_{i_1\cdots i_{10}}, \tag{3.18}$$

which satisfies

$$(\Gamma^{(0)})^2 = \mathbb{1}. \tag{3.19}$$

In (3.17) we have used the property

$$\Gamma^{(0)}\gamma^{l_1\cdots l_k} = \frac{(-)^{k(k-1)/2}}{(10-k)!\sqrt{-\det g}}\,\epsilon^{i_1\cdots i_{10-k}l_1\cdots l_k}\gamma_{i_1\cdots i_{10-k}} \tag{3.20}$$

for $k=1$. From (3.17) we read off that

$$\Gamma_0 = \Gamma^{(0)}\sigma_1, \quad \mathcal{T}_0 = \gamma^i\partial_i\theta. \tag{3.21}$$

Obviously $\Gamma^2 = \mathbb{1}$ to this order.

So the Born–Infeld term should vary into $\mathcal{T}$, which is indeed achieved by setting

$$\mathcal{L}_{BI0} = -\sqrt{-\det\eta}\,(1+\tfrac{1}{2}\overline{\theta}\gamma^i\partial_i\theta). \tag{3.22}$$

This gives

$$\delta\mathcal{L}_{BI0} = -\sqrt{-\det\eta}\,\overline{\eta}\gamma^i\partial_i\theta, \tag{3.23}$$

where we have used the variation of X^μ as given in (3.1).

A similar analysis can be done for the terms of higher order in F. At the linear level the variation of the Wess–Zumino term is

$$\delta\mathcal{L}_{WZ1}=\tfrac{1}{2}\sqrt{-\det\eta}\,\overline{\eta}(i\sigma_2)\Gamma^{(0)}(\gamma^{jk}F_{jk}\gamma^i\partial_i\theta-2\gamma_i\partial_j\theta F^{ij}). \tag{3.24}$$

The variation of the complete action should be

$$\delta\mathcal{L}_1=-\sqrt{-\det\eta}\,\overline{\eta}\{\mathcal{T}_1-(\Gamma_0\mathcal{T}_1+\Gamma_1\mathcal{T}_0)\}. \tag{3.25}$$

So we read off that

$$\Gamma_1=\Gamma^{(0)}(i\sigma_2)\tfrac{1}{2}\gamma^{kl}F_{kl}, \quad \mathcal{T}_1=\sigma_3\gamma_i\partial_j\theta F^{ij}. \tag{3.26}$$

Note that $\Gamma^2=\mathbb{1}$ at this order in F because

$$\{\sigma_1,\Gamma_1\}=0. \tag{3.27}$$

This is a general feature: the condition that $\Gamma^2=\mathbb{1}$ only contains useful information at even orders in the expansion in F. At odd orders it is just a consequence of the properties of $\mathcal{P}_{(k)}$.

The variation under supersymmetry and κ-symmetry of the term linear in F in the Born–Infeld action (3.8) reads

$$\delta\mathcal{L}_{BI1}=-\sqrt{-\det\eta}\,\overline{\eta}\sigma_3\gamma_i\partial_j\theta F^{ij}-\tfrac{1}{2}(\overline{\epsilon}+\overline{\eta})\gamma_i\theta\partial_j\{\sqrt{-\det\eta}F^{ij}\}. \tag{3.28}$$

The first term is the required contribution of $\mathcal{T}_1$. The second term must be cancelled by the variation of V in the F^2 term. The F^2 term gives

$$-\delta V_i\partial_j\{\sqrt{-\det\eta}F^{ij}\}, \tag{3.29}$$

which implies the following variation of V_i:

$$\delta V_i=-\tfrac{1}{2}(\overline{\epsilon}+\overline{\eta})\sigma_3\gamma_i\theta. \tag{3.30}$$

Therefore the combination

$$\mathcal{F}_{ij}=F_{ij}+\overline{\theta}\sigma_3\gamma_{[i}\partial_{j]}\theta \tag{3.31}$$

is supersymmetric and transforms covariantly under κ-symmetry.

At the quadratic level we get

$$\delta\mathcal{L}_{WZ2}=\tfrac{1}{8}\sqrt{-\det\eta}\,\overline{\eta}\sigma_1\Gamma^{(0)}(\gamma_{ijkl}\gamma^m\partial_m\theta-4\gamma_{ijk}\partial_l\theta)F^{[ij}F^{kl]}. \tag{3.32}$$

The order 2 terms in the variation of the total action are

$$\delta\mathcal{L}_2=-\sqrt{-\det\eta}\,\overline{\eta}(-(\Gamma_2\gamma^i\partial_i\theta+\Gamma_1\mathcal{T}_1+\sigma_1\mathcal{T}_2)+\mathbb{1}\mathcal{T}_2), \tag{3.33}$$

where Γ_1 and $\mathcal{T}_1$ were determined at the linear level. On the other hand, from $\Gamma^2=\mathbb{1}$ we have

$$\sigma_1\Gamma_2+\Gamma_2\sigma_1+\Gamma_1\Gamma_1=0, \tag{3.34}$$

from which we obtain (using $\Gamma_2\sim\sigma_1$)

$$\Gamma_2=-\tfrac{1}{2}\sigma_1\Gamma_1\Gamma_1=\Gamma^{(0)}\sigma_1\{\tfrac{1}{8}\gamma_{ijkl}F^{ij}F^{kl}-\tfrac{1}{4}F^{kl}F_{kl}\}. \tag{3.35}$$

Substituting all this in (3.33), we find that

$$\mathcal{T}_2 = \gamma_i \partial_j \theta (F^{ik} F_k^j + \tfrac{1}{4} \eta^{ij} F^{kl} F_{kl}). \tag{3.36}$$

This indeed agrees with the variation of the Born–Infeld action.

There is a feature about the Abelian case which just starts being visible in the quadratic terms. It is obviously possible to write Γ at this order in the form

$$\Gamma = (1 - \tfrac{1}{4} F^{kl} F_{kl}) \Gamma^{(0)} ((\sigma_1 + \tfrac{1}{2}(i\sigma_2)\gamma^{kl} F_{kl} + \tfrac{1}{8} \sigma_1 \gamma_{ijkl} F^{ij} F^{kl}) \tag{3.37}$$

up to terms of higher order in F. In fact, this factorization is a general feature of the Abelian action which is also valid for the complete answer,

$$\Gamma = \frac{\sqrt{-\det g}}{\sqrt{-\det(g+\mathcal{F})}} \Gamma^{(0)} \sum_{k=0}^{5} \frac{1}{2^k k!} \mathcal{P}_{(k)} \gamma^{i_1 \cdots i_{2k}} \mathcal{F}^k_{i_1 \cdots i_{2k}}. \tag{3.38}$$

The iterative procedure will obviously confirm this factorization, as is shown by continuing to higher orders in F. However, in this construction it is not clear why the factorization should occur. This is an issue in the non-Abelian situation where the complete answer is not known. Note that the factor

$$\frac{\sqrt{-\det g}}{\sqrt{-\det(g+\mathcal{F})}} \tag{3.39}$$

in (3.38) contains the inverse of the Born–Infeld action. The idea that Γ provides the explicit form of the Born–Infeld action was part of our motivation to use κ-symmetry as a means of constructing the non-Abelian Born–Infeld action.

From (3.36) it is clear that $\mathcal{T}$, at least at the quadratic level, shows a similar factorization property as Γ. The full answer for $\mathcal{T}$ is of the form,

$$\mathcal{T} = \sqrt{-\det(g+\mathcal{F})} \sum_{k=0}^{\infty} (\sigma_3)^k \gamma_i \partial_j \theta (\mathcal{F}^k)^{ij}, \tag{3.40}$$

where

$$(\mathcal{F}^k)^{ij} = \mathcal{F}^{il_1} \mathcal{F}_{l_1 l_2} \cdots \mathcal{F}^{l_{k-1} j} \quad k>0,$$

$$(\mathcal{F}^k)^{ij} = g^{ij} \quad k=0. \tag{3.41}$$

We will see a similar feature in the non-Abelian case.

The Abelian case discussed above can easily be generalized to $U(1)^n$. Then the κ-symmetric action is just the sum of n actions of the type discussed in this section. For n overlapping branes one would need only one set of embedding coordinates X^μ to describe this truncation of the non-Abelian situation. This would be similar to treating X^μ as a singlet of $U(n)$ in the non-Abelian case. Note however that this sum of actions is very different from a single Born–Infeld action with world volume metric,

$$g_{ij} = \eta_{ij} + \overline{\theta^A} \gamma_{(i} \partial_{j)} \theta^A, \tag{3.42}$$

summed over the $nU(n)$ branes. A metric (3.42) would be like taking the trace *inside* the root of the Born–Infeld term, while it is known from open string amplitude calculations that there should be a single trace (with some ordening prescription) which produces in the $U(1)^n$ case a sum of separate Born–Infeld terms.

IV. THE NON-ABELIAN BORN–INFELD ACTION

For the non-Abelian case we start with the following Wess–Zumino term:

$$\mathcal{L}_{WZ}=\epsilon^{i_1\cdots i_{10}}\sum_{k=0}^{4}\frac{(-1)^k}{2^{k+1}k!(9-2k)!}\overline{\theta}^A\mathcal{P}_{(k)}^{ABC_1\cdots C_k}\gamma_{i_1\cdots i_{9-2k}}\mathcal{D}_{i_{10-2k}}\theta^B F^{C_1}_{i_{11-2k}i_{12-2k}}\cdots F^{C_k}_{i_9 i_{10}}. \tag{4.1}$$

The tensors $\mathcal{P}$ are symmetric in the indices C_i contracted with F. $\mathcal{P}$ also contains the Pauli matrices to specify the $N=2$ structure for the fermions. We have the following possibilites:

$$\begin{array}{lll} k \text{ even:} & \mathbb{1}_2,\sigma_1,\ \sigma_3 & \text{symmetry in } AB,\\ & i\sigma_2 & \text{antisymmetry in } AB,\\ k \text{ odd:} & \mathbb{1}_2,\ \sigma_1,\ \sigma_3 & \text{antisymmetry in } AB,\\ & i\sigma_2 & \text{symmetry in } AB. \end{array} \tag{4.2}$$

This requirement follows from the fact that the bilinear fermions in the action (4.1) should not be a total derivative. The Yang–Mills structure of $\mathcal{P}$ arises from the trace of $k+2$ generators in the fundamental representation of $\mathrm{U}(n)$, and will be built from the structure constant f_{ABC} (completely antisymmetric) and from the completely symmetric tensors d_{ABC}. In the Appendix we gather useful properties of these tensors.

The general form of (4.1) follows from the requirement that the Wess–Zumino term is of topological nature, i.e., independent of the metric. The coefficients have been chosen equal to those in the Abelian case, which amounts to a particular normalization of the $\mathcal{P}_{(k)}$. Note that we do not assume a particular ordening in the trace, i.e., there is no a priori symmetry imposed between the indices C_i on the one hand, and A, B on the other hand.

So at order 0 we start with

$$\mathcal{P}_{(0)}^{AB}=\sigma_1\,\mathrm{tr}\,T^AT^B=\sigma_1\delta^{AB}, \tag{4.3}$$

and the variation of the lowest order contribution in (4.1) is

$$\delta\mathcal{L}_{WZ0}=\sqrt{-\det\eta}\,\overline{\eta}^A\sigma_1\Gamma^{(0)}\gamma^i\mathcal{D}_i\theta^A. \tag{4.4}$$

We write the variation of the complete action at this order as

$$\delta\mathcal{L}_0=-\sqrt{-\det\eta}\,\overline{\eta}^A(\delta^{AB}-\Gamma_0^{AB})\mathcal{T}_0^{B}, \tag{4.5}$$

so that

$$\Gamma_0^{AB}=\Gamma^{(0)}\delta^{AB}\sigma_1,\quad \mathcal{T}_0^A=\gamma^i\mathcal{D}_i\theta^A. \tag{4.6}$$

Cleary $\Gamma^{AC}\Gamma^{CB}=\delta^{AB}$. In the above we considered X^μ to be a singlet under $\mathrm{U}(n)$ transformations. To go to the static gauge we would have to set $\det\eta\rightarrow -1$.

The Born–Infeld term must reproduce the first term in (4.5). The only choice is to have

$$\mathcal{L}_{BI}=-\sqrt{-\det\eta}\,(1+\tfrac{1}{2}\overline{\theta}^A\gamma^i\mathcal{D}_i\theta^A). \tag{4.7}$$

We indeed find the correct variation if we set

$$\delta X^\mu=\tfrac{1}{2}\overline{\epsilon}^A\Gamma^\mu\theta^A+\tfrac{1}{2}\overline{\eta}^A\Gamma^\mu\theta^A. \tag{4.8}$$

With this choice of δX the metric (3.42) becomes supersymmetric and covariant under κ-transformations. As we explained at the end of Sec. III this is not the natural metric for the non-Abelian [or for the $\mathrm{U}(1)^n$] situation. So the different choices for X^μ start to diverge at this point. At the linear level there is still no crucial difference between the choice of a singlet X^{μ} and the static gauge. At the quadratic level, however, the singlet choice will fail.

The variation of the linear contribution to the Wess–Zumino term gives

$$\delta\mathcal{L}_{WZ1}=\tfrac{1}{2}\sqrt{-\det\eta}\,\overline{\eta}^{A}\mathcal{P}^{ABC}_{(1)}\Gamma^{(0)}(\gamma_{ij}\gamma^{k}\mathcal{D}_{k}\theta^{B}-2\gamma_{i}\mathcal{D}_{j}\theta^{B})F^{Cij}. \tag{4.9}$$

The first term must correspond to $\Gamma_{1}\mathcal{T}_{0}$, from which we read off that

$$\Gamma_{1}^{AB}=\Gamma^{(0)}\mathcal{P}^{ABC}_{(1)}\,\tfrac{1}{2}\,\gamma_{ij}F^{ijC}. \tag{4.10}$$

From $\Gamma^{2}=\mathbb{1}$ at linear order we find

$$\{\sigma_{1},\Gamma_{1}^{AB}\}=0, \tag{4.11}$$

so that $\mathcal{P}_{(1)}$ must have the following form:

$$\mathcal{P}^{ABC}_{(1)}=(i\sigma_{2})d^{ABC}+c_{1}\sigma_{3}f^{ABC}. \tag{4.12}$$

The coefficient of the *d*-term is chosen to agree with the Abelian case. The coefficient of the *f*-term is arbitrary, and with a field redefinition

$$\theta^{A}\rightarrow\theta^{A}-\tfrac{1}{4}c_{1}(i\sigma_{2})f^{ABC}\gamma^{kl}\theta^{B}F^{C}_{kl}, \tag{4.13}$$

and a corresponding redefinition of the vector field, the *f*-term can be eliminated. This is the choice we will make, because then we stay as close as possible to the Abelian situation.

In fact, the whole term linear in F in the Wess–Zumino action can be transformed away by a field redefinition, also in the Abelian case. It is not surprising that these linear terms can be eliminated, since they are part of the supersymmetrization of the bosonic F^{3} term, which we know to be absent. The reason we will keep the usual linear term is that in this form the answer in the Abelian case takes on a relatively simple form.[11]

We also find

$$\mathcal{T}_{1}^{A}=-\sigma_{1}\mathcal{P}^{ABC}_{(1)}\gamma_{i}\mathcal{D}_{j}\theta^{B}F^{ijC}, \tag{4.14}$$

so that in analogy with the Abelian case the Born–Infeld term must contain

$$\mathcal{L}_{BI1}=-\sqrt{-\det\eta}\left(-\tfrac{1}{2}\overline{\theta}^{A}\sigma_{1}\mathcal{P}^{ABC}_{(1)}\gamma_{[i}\mathcal{D}_{j]}\theta^{B}F^{ijC}+\tfrac{1}{4}F^{ijA}F^{A}_{ij}\right). \tag{4.15}$$

The variation of this term reproduces correctly the $\mathcal{T}_{1}$ contribution, and the remainder is cancelled by introducing a variation of V_{i},

$$\delta V_{i}^{C}=+\tfrac{1}{2}(\overline{\epsilon}^{A}+\overline{\eta}^{A})\sigma_{(1)}\mathcal{P}^{ABC}_{(1)}\gamma_{i}\theta^{B}. \tag{4.16}$$

We can then define a supersymmetric and κ-covariant $\mathcal{F}^{C}$ as

$$\mathcal{F}^{C}_{ij}=F^{C}_{ij}-\overline{\theta}^{A}\sigma_{1}\mathcal{P}^{ABC}_{(1)}\gamma_{[i}\mathcal{D}_{j]}\theta^{B}. \tag{4.17}$$

This defines the non-Abelian generalization of the NS–NS two-form field, to this order.

At the quadratic level things become more complicated. The variation of the Wess–Zumino term gives

$$\delta\mathcal{L}_{WZ2}=\sqrt{-\det\eta}\,\overline{\eta}^{A}\mathcal{P}^{ABCD}_{(2)}\Gamma^{(0)}\{\tfrac{1}{8}\gamma_{ijkl}\gamma^{m}\mathcal{D}_{m}\theta^{B}-\tfrac{1}{2}\gamma_{ijk}\mathcal{D}_{l}\theta^{B}\}F^{Cij}F^{Dkl}. \tag{4.18}$$

The variations quadratic in F will have to generate the following contributions:

$$\delta\mathcal{L}_2 = -\sqrt{-\det\eta}\,\overline{\eta}^A(-(\Gamma_2^{AB}\gamma^i\mathcal{D}_i\theta^B + \Gamma_1^{AB}\mathcal{T}_1^B + \Gamma_0^{AB}\mathcal{T}_2^B) + \mathbb{1}\mathcal{T}_2^A). \tag{4.19}$$

We can determine Γ_2 from the requirement that $\Gamma^2 = \mathbb{1}$ at order 2,

$$\sigma_1\Gamma_2^{AB} + \Gamma_2^{AB}\sigma_1 + \Gamma_1^{AC}\Gamma_1^{CB} = 0. \tag{4.20}$$

This calculation assumes that $[\Gamma_2, \sigma_1] = 0$, and requires the product of the $\mathcal{P}_{(1)}$-tensors. The result is

$$\Gamma_2^{AB} = -\Gamma^{(0)}\sigma_1(\mathcal{S}^{ABCD}(\tfrac{1}{8}\gamma_{ijkl}F^{ijC}F^{klD} - \tfrac{1}{4}F^{C}_{kl}F^{klD}) + \mathcal{A}^{ABCD}\tfrac{1}{2}\gamma_{ij}F^{ikC}F_k^{jD}). \tag{4.21}$$

Here we have defined

$$\mathcal{S}^{ABCD} \equiv \mathcal{P}_{(1)}^{AE(C}\mathcal{P}_{(1)}^{BD)E} = -d^{AE(C}d^{BD)E}, \tag{4.22}$$

$$\mathcal{A}^{ABCD} \equiv \mathcal{P}_{(1)}^{AE[C}\mathcal{P}_{(1)}^{BD]E} = -d^{AE[C}d^{BD]E}, \tag{4.23}$$

where the (anti-)symmetrization is over the indices C and D only. Note that tensors $\mathcal{S}^{ABCD}$ and $\mathcal{A}^{ABCD}$ are then symmetric and antisymmetric, resp., in the index pairs AB and CD.

To solve the remainder of (4.19) we have to make the choice

$$\mathcal{P}_{(2)}^{ABCD} = -\sigma_1\mathcal{S}^{ABCD}. \tag{4.24}$$

We then find the following result for $\mathcal{T}_2$:

$$\mathcal{T}_2^A = -\mathcal{S}^{ABCD}\gamma_{(i}\mathcal{D}_{j)}\theta^B(F^{ikC}F_k^{jD} + \tfrac{1}{4}\eta^{ij}F^C_{kl}F^{klD}) + \tfrac{1}{2}\mathcal{A}^{ABCD}\gamma_{ijk}\{\mathcal{D}^k\theta^BF^{ilC}F_l^{jD} - \mathcal{D}_l\theta^BF^{ijC}F^{klD}\}. \tag{4.25}$$

The result (4.25) agrees with the Abelian result (3.36) if we truncate from U(n) to U(1).

The Born–Infeld term now has to reproduce $\mathcal{T}_2$, while remaining contributions may be cancelled by introducing an additional variation of V_i^A. It is at this stage that the choice of X^μ as a Yang–Mills singlet runs into trouble. Contributions to this variation of the Born–Infeld term come from the F^2 term in the action, when the metric η_{ij}, depending on X^μ, is varied. Such variations contain a double sum over U(n) indices, i.e., they would be of the form [using (4.8)]

$$\partial_iX^\mu\partial_j(\eta^A\Gamma_\mu\theta^A)F^{Cik}F^Ck^j. \tag{4.26}$$

Such terms would have the wrong U$(1)^n$ limit, and cannot be canceled by other contributions. Partial integration does not help, since it produces symmetric second derivatives on X^μ, which do not occur elsewhere. It is at this stage that we should say farewell to the embedding coordinates X^μ, and proceed in the static gauge.

Terms in the Born–Infeld action that might play a role in this analysis are

$$\begin{aligned}\mathcal{L}_{BI2} = -(&\tfrac{1}{4}F^{ijC}F^C_{ij} + \alpha F^{ikA}F_k^{jB}F^C_{ji}f^{ABC} - \tfrac{1}{2}\overline{\theta}^A\mathcal{S}^{ABCD}\gamma_{(i}\mathcal{D}_{j)}\theta^B\{F^{ikC}F_k^{jD} + \tfrac{1}{4}\eta^{ij}F^C_{kl}F^{klD}\}\\ &+\tfrac{1}{4}\overline{\theta}^A\mathcal{A}^{ABCD}\gamma_{ijk}\{\mathcal{D}^k\theta^BF^{ilC}F_l^{jD} - \mathcal{D}_l\theta^BF^{ijC}F^{klD}\}).\end{aligned} \tag{4.27}$$

Note that an F^3 term is in principle not excluded in the non-Abelian case.

In the static gauge we know how to deal with these terms. Then we need not vary the F^2 term. In the F^3 term we have to vary F, which gives an F^2 variation with a single γ-matrix. Therefore it does not relate to the $\mathcal{A}$-terms, which have three γ-matrices, and we must choose α equal to zero. In the $\mathcal{S}$-and $\mathcal{A}$-terms we can perform partial integrations to get rid of the ϵ and $\partial\eta$ terms in the variation. These give equations of motion of V, and can be cancelled by new variations of V. The required identities are

$$\mathcal{D}_j(F^{ik(C}F_k^{jD)}+\tfrac{1}{4}\eta^{ij}F_{kl}^{(C}F^{klD)})=F^{ik(C}\mathcal{D}_jF_k^{jD)}, \tag{4.28}$$

$$\mathcal{D}_{[k}(F_i^{l[C}F_{ij]}^{D]})-\mathcal{D}_l(F_{[ij}^{[C}F_{k]}^{lD]})=-F_{[ij}^{[C}\mathcal{D}_lF_{k]}^{lD]}. \tag{4.29}$$

The remaining η terms in the variation give us precisely $\mathcal{T}_2$. This leads to the following new variations of V_i^A,

$$\delta V_i^A=+\tfrac{1}{2}(\overline{\epsilon}^B+\overline{\eta}^B)\mathcal{S}^{BCDA}\gamma_k\theta^CF^{kiD}+\tfrac{1}{4}(\overline{\epsilon}^B+\overline{\eta}^B)\mathcal{A}^{BCDA}\gamma_{ikl}\theta^CF^{klD}. \tag{4.30}$$

Note that the variation of V_i^A no longer agrees with the result given in (3.30). However, now we should compare with the Abelian result in static gauge. This gauge choice requires a compensating world volume coordinate transformation, which, when acting on V_i, produces the Abelian limit of the $\mathcal{S}$-contribution in (4.30). The $\mathcal{A}$-term in (4.30) vanishes in the Abelian limit.

V. SUMMARY

In this section we will summarize the results obtained in the non-Abelian case. The action is the sum of the Born–Infeld and Wess–Zumino terms. The Wess–Zumino term looks as follows:

$$\begin{aligned}\mathcal{L}_{WZ}=\epsilon^{i_1\cdots i_{10}}\bigg\{&\frac{1}{2\cdot 9!}\overline{\theta}^A\sigma_1\gamma_{i_1\cdots i_9}\mathcal{D}_{i_{10}}\theta^A-\frac{1}{4\cdot 7!}\overline{\theta}^A\mathcal{P}_{(1)}^{ABC}\gamma_{i_1\cdots i_7}\mathcal{D}_{i_8}\theta^BF^C_{i_9i_{10}}\\&+\frac{1}{16\cdot 5!}\overline{\theta}^A(-\sigma_1\mathcal{S}^{ABCD})\gamma_{i_1\cdots i_5}\mathcal{D}_{i_6}\theta^B(F^CF^D)_{i_7\cdots i_{10}}\bigg\}.\end{aligned} \tag{5.1}$$

The Born–Infeld action is

$$\begin{aligned}\mathcal{L}_{BI}=-\{&1+\tfrac{1}{2}\overline{\theta}^A\gamma^i\mathcal{D}_i\theta^A-\tfrac{1}{2}\overline{\theta}^A\sigma_1\mathcal{P}_{(1)}^{ABC}\gamma_{[i}\mathcal{D}_{j]}\theta^BF^{ijC}+\tfrac{1}{4}F^{ijA}F_{ij}^A-\tfrac{1}{2}\overline{\theta}^A\mathcal{S}_{ABCD}\gamma_{(i}\mathcal{D}_{j)}\theta^B\{F^{ikC}F_k^{jD}\\&+\tfrac{1}{4}\eta^{ij}F_{kl}^CF^{klD}\}+\tfrac{1}{4}\overline{\theta}^A\mathcal{A}^{ABCD}\gamma_{ijk}\{\mathcal{D}^k\theta^BF^{ilC}F_l^{jD}-\mathcal{D}_l\theta^BF^{ijC}F^{klD}\}\}.\end{aligned} \tag{5.2}$$

In the action we use the following Yang–Mills structures:

$$\mathcal{P}_{(1)}^{ABC}=(i\sigma_2)d^{ABC}, \tag{5.3}$$

$$\mathcal{S}^{ABCD}=\mathcal{P}_{(1)}^{AE(C}\mathcal{P}_{(1)}^{BD)E}=-d^{AE(C}d^{BD)E}, \tag{5.4}$$

$$\mathcal{A}^{ABCD}=\mathcal{P}_{(1)}^{AE[C}\mathcal{P}_{(1)}^{BD]E}=-d^{AE[C}d^{BD]E}. \tag{5.5}$$

The action is invariant under global supersymmetry transformations and local κ-transformations,

$$\delta\overline{\theta}^A=-\overline{\epsilon}^A+\overline{\eta}^A, \tag{5.6}$$

$$\delta V_i^A=\tfrac{1}{2}(\overline{\epsilon}^B+\overline{\eta}^B)\sigma_1\mathcal{P}_{(1)}^{BCA}\gamma_i\theta^C+\tfrac{1}{2}(\overline{\epsilon}^B+\overline{\eta}^B)\mathcal{S}^{BCDA}\gamma_k\theta^CF^{kiD}+\tfrac{1}{4}(\overline{\epsilon}^B+\overline{\eta}^B)\mathcal{A}^{BCDA}\gamma_{ikl}\theta^CF^{klD}, \tag{5.7}$$

where the parameters ϵ^A satisfy the condition,

$$f_{ABC}\epsilon^C=0. \tag{5.8}$$

As explained in the previous sections, the variation of the action under κ-symmetry can be expressed in terms of

$$\begin{aligned}\Gamma^{AB}=\Gamma^{(0)}\{&\sigma_1\delta^{AB}+\mathcal{P}_{(1)}^{ABC}\tfrac{1}{2}\gamma^{kl}F_{kl}^C-\sigma_1\mathcal{S}^{ABCD}(\tfrac{1}{8}\gamma_{ijkl}F^{ijC}F^{klD}-\tfrac{1}{4}F_{kl}^CF^{klD})\\&-\sigma_1\mathcal{A}^{ABCD}\tfrac{1}{2}\gamma_{ij}F^{ikC}F_k^{jD}\},\end{aligned} \tag{5.9}$$

$$\mathcal{T}^A=\gamma^i\mathcal{D}_i\theta^A-\sigma_1\mathcal{P}_{(1)}^{\ ABC}\gamma_i\mathcal{D}_j\theta^B F^{ijC}-\mathcal{S}^{\ ABCD}\gamma_{(i}\mathcal{D}_{j)}\theta^B(F^{ikC}F_k^{jD}+\tfrac{1}{4}\eta^{ij}F_{kl}^{C}F^{klD})$$
$$+\tfrac{1}{2}\mathcal{A}^{ABCD}\gamma_{ijk}\{\mathcal{D}^k\theta^B F^{ilC}F_l^{jD}-\mathcal{D}_l\theta^B F^{ijC}F^{klD}\}. \tag{5.10}$$

VI. GAUGE FIXING

In the κ-symmetric system which we obtained in this paper, the ordinary supersymmetry is hidden in the local κ-symmetry, and to make it explicit, κ-symmetry should be gauge fixed. This analysis is very similar to the one done in the Abelian case in Ref. 11.

To do this analysis it is convenient to write out the $N=2$ doublets explicitly. We write

$$\Gamma=\begin{pmatrix} 0 & \gamma \\ \tilde{\gamma} & 0 \end{pmatrix}, \tag{6.1}$$

with

$$\Gamma^2=\begin{pmatrix} \gamma\tilde{\gamma} & 0 \\ 0 & \tilde{\gamma}\gamma \end{pmatrix}=\mathbb{1}. \tag{6.2}$$

Here γ, $\tilde{\gamma}$ are 32×32 matrices, with in addition indices AB, where A, B run from 1 to n^2. Then, splitting also the fermions into separate $N=1$ fermions, we write variations as follows:

$$\delta\overline{\theta_1^A}=-\overline{\epsilon_1^A}+\overline{\eta_1^A}, \quad \delta\overline{\theta_2^A}=-\overline{\epsilon_2^A}+\overline{\eta_2^A}. \tag{6.3}$$

The parameters η can be expressed in terms of parameters κ,

$$\overline{\eta}=(\overline{\eta_1}\quad\overline{\eta_2})=(\overline{\kappa_1}+\overline{\kappa_2}\tilde{\gamma}\quad\overline{\kappa_2}+\overline{\kappa_1}\gamma). \tag{6.4}$$

Now we choose a κ-gauge by setting $\overline{\theta_2}=0$, which implies that the transformation parameters must satisfy

$$\overline{\kappa_2}=\overline{\epsilon_2}-\overline{\kappa_1}\gamma. \tag{6.5}$$

So after κ-gauge fixing the remaining η is

$$\overline{\eta_1}=\overline{\epsilon_2}\tilde{\gamma}, \tag{6.6}$$

and the remaining fermions $\chi^A\equiv\theta_1^A$ transform as

$$\delta\overline{\chi^A}=-\overline{\epsilon_1^A}+\overline{\epsilon_2^B}\tilde{\gamma}^{BA}. \tag{6.7}$$

Let us now look at the gauge fixed action. The Wess–Zumino term vanishes after gauge fixing, since it was off-diagonal in the fermions θ_1 and θ_2. The Born–Infeld term gives

$$\mathcal{L}_{BI}=-\{1+\tfrac{1}{2}\overline{\chi^A}\gamma^i\mathcal{D}_i\chi^A+\tfrac{1}{2}d_{ABC}\overline{\chi^A}\gamma_{[i}\mathcal{D}_{j]}\chi^B F^{ijC}+\tfrac{1}{4}F^{ijA}F_{ij}^{A}+\tfrac{1}{2}d^{AEC}d^{BDE}\overline{\chi^A}\gamma_{(i}\mathcal{D}_{j)}\chi^B\{F^{ikC}F_k^{jD}$$
$$+\tfrac{1}{4}\eta^{ij}F_{kl}^{C}F^{klD}\}-\tfrac{1}{4}d^{AE[C}d^{BD]E}\overline{\chi^A}\gamma_{ijk}\{\mathcal{D}^k\chi^B F^{ilC}F_l^{jD}-\mathcal{D}_l\chi^B F^{ijC}F^{klD}\}\}. \tag{6.8}$$

Note that the terms of the form $\overline{\chi}\partial\chi F^2$ are not symmetric traces of $\mathrm{U}(n)$ generators. The symmetric trace is of the form,

$$\mathrm{tr}\, T_{(A}T_BT_CT_{D)}=\tfrac{1}{3}(d_{ABE}d_{CDE}+d_{CAE}d_{BDE}+d_{BCE}d_{ADE}). \tag{6.9}$$

The second line in (6.8) contains only two of the three contributions needed for the symmetric trace, the last line contains explicit antisymmetrizations and can be rewritten in terms of structure constants,

$$d_{AEC}d_{BDE}-d_{AED}d_{BCE}=f_{ABE}f_{CDE}\,. \tag{6.10}$$

Finally, the linear and nonlinear supersymmetry transformations which leave (6.8) invariant are of the form,

$$\begin{aligned}\delta\chi^A&=-\overline{\epsilon}_1^A-\overline{\epsilon}_2^A+\overline{\epsilon}_2^B\{d^{BAC}\tfrac{1}{2}\gamma^{kl}F_{kl}^C+\mathcal{S}^{BACD}(\tfrac{1}{8}\gamma_{ijkl}F^{ijC}F^{klD}-\tfrac{1}{4}F_{kl}^CF^{klD})\},\\ \delta V_i^A&=-\tfrac{1}{2}(\overline{\epsilon}_1^B-\overline{\epsilon}_2^B)d^{BCA}\gamma_i\chi^C-\tfrac{1}{4}\overline{\epsilon}_2^Bd^{BED}d^{ECA}\gamma_{kl}\gamma_i\chi^CF^{klD}+\tfrac{1}{2}(\overline{\epsilon}_1^B-\overline{\epsilon}_2^B)\mathcal{S}^{BCDA}\gamma_k\chi^CF^{kiD}.\end{aligned} \tag{6.11}$$

Note that contributions with $\mathcal{A}$, the antisymmetrized product of two d-tensors, do not appear because of (6.10) and the fact that ϵ must be in the U(1) direction.

VII. CONCLUSIONS

In this paper we have obtained the non-Abelian generalization of the Born–Infeld action up to terms quartic in the Yang–Mills field strength, and including all fermion bilinear terms up to terms cubic in the field strength. The terms of the form $\overline{\chi}\partial\chi F^2$ violate the symmetric trace conjecture.

The κ-symmetric construction of the Born–Infeld action involves the matrix Γ, satisfying $\Gamma^2=\mathbb{1}$, which is used to project away one of the components of the fermion doublet. In the Abelian case Γ factorizes in a part that is polynomial in F, and the inverse of the Born–Infeld action, which expands to an infinite series in F. In the non-Abelian case we are not yet at a stage that such a factorization could be recognized. We see however, that the result (5.9) is consistent with a factorization of the form,

$$\begin{aligned}\Gamma^{AB}=\Gamma^{(0)}\{&\sigma_1\delta^{AC}+\mathcal{P}_{(1)}^{ACF}\tfrac{1}{2}\gamma^{kl}F_{kl}^F-\sigma_1\mathcal{S}^{ACFG}(\tfrac{1}{8}\gamma_{ijkl}F^{ijF}F^{klG})-\sigma_1\mathcal{A}^{ACFG}\tfrac{1}{2}\gamma_{ij}F^{ikF}F_k^{jG}\}\\ &\times(\delta^{CB}+\mathcal{S}^{CBDE}\tfrac{1}{4}F_{mn}^DF^{mnE}).\end{aligned} \tag{7.1}$$

Note that, as in the Abelian case, $\mathcal{T}$ (5.10) contains the inverse of the factor that we find in Γ. Clearly the second factor in the above expression is not a U(n) singlet, and therefore does not correspond to the inverse of the action. Further analysis, which we plan to do at the cubic and quartic level in F, should elucidate in which sense these factors are related to the Born–Infeld action.

It is intriguing that κ-symmetry and worldvolume reparametrization invariance appear to be incompatible. Although for applications such as the construction of non-Abelian BPS states this is not a drawback, issues of superspace and curved background remain unclear in the static gauge. One way to try to resolve the issue of embedding coordinates would be to look in more detail at the transformation rules of V_i^A. If a formulation with world volume reparametrization invariance exists, then our formulation should be its gauge fixed version, and we should recognize the corresponding compensating transformation in the transformation rule of V_i^A. To give an example, let us consider the possibility that the embedding coordinates are in the adjoint of U(n), and transform as

$$\delta X^{\mu A}=d^{ABC}\xi^{iB}\partial_iX^{\mu C}+\delta_\epsilon X^{\mu A}+\delta_\eta X^{\mu A}. \tag{7.2}$$

Let us now gauge fix this extended worldvolume symmetry, by setting $X^A=0$ for all $A\in\mathrm{SU}(n)$, and $X^{\mu 1}=\delta_i^\mu\sigma_i$ for $A=1\in\mathrm{U}(1)$. Then the compensating transformation which preserves this gauge is of the form,

$$\xi^{\mu A}=-\delta_\epsilon X^{\mu A}-\delta_\eta X^{\mu A}. \tag{7.3}$$

Here we have used that $d^{AB1}\sim\delta^{AB}$, in a basis where the U(1) component of U(n) is labeled by $A=1$. Then, if the variation of V_i^A under world volume coordinate transformations is of the form,

$$\delta V_i^A = d^{ABC} \xi^{kB} F_{ki}^C , \tag{7.4}$$

we see that the term proportional to $\mathcal{S}$ in (5.7) has almost the right form to be interpreted as a compensating transformation. However, the Yang–Mills structure in this expression is not quite correct, as the indices of V and F are not on the same d-tensor. We have found that if one chooses in (4.12) $c_1 = 1$, that then the structure comes out all right, and gives

$$\delta X^{\mu A} \sim d^{ABC} d^{BDE} (\bar{\epsilon}^D + \bar{\eta}^D) \gamma^k \theta^E \partial_k X^{\mu C}. \tag{7.5}$$

It would be interesting to see whether or not the construction of the generalized Born–Infeld action including the worldvolume structure indicated above is possible.

As mentioned above, the generalization to a curved background would be greatly facilitated by a better understanding of the superspace structure of the D-brane action. However, there is also another open issue to consider. Consider the expression (4.17), where we give the non-Abelian generalization of the relation $\mathcal{F} = F + B$ in a flat background. In going to a curved background we have to decide how and where to introduce the NS–NS fields. Should there be a non-Abelian generalization of the NS–NS field B, or do only the U(1) fields on the worldvolume couple effectively to the background fields? Similar questions can be raised about the RR-fields [see (4.1)], whose form in a flat background also suggests that a non-Abelian generalization should be required.

In a future publication,[20] we hope to extend this work to higher order in F, and to apply the results to the construction of non-Abelian BPS states. The simplest situation to think of is the relation between D-branes at angles[21] and overlapping branes through T-duality.[7] As was shown in Ref. 8 the BPS conditions between angles translate to conditions between magnetic fields F which include contributions cubic in F. Therefore we will have to go at least to order F^3 in the supersymmetry transformation rules to be able to compare our results with the predictions implied by Ref. 21. In the Abelian case the relation between κ-symmetric formulations and BPS states was formulated in Ref. 22. In particular, there it was shown that the knowledge of Γ is in fact sufficient to obtain BPS states. It would be interesting to generalize these results to the non-Abelian situation.

Finally, it would be instructive to apply other approaches than the one employed in this paper to find the complete answer. For instance, one could use the superembedding techniques developed in Refs. 23 and 24. Yet another approach could be to extend to the non-Abelian case the analysis of Ref. 25, where it was shown how the super world volume dynamics of superbranes can be obtained from nonlinear realizations.

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APPENDIX: PROPERTIES OF U(n) GENERATORS, ETC.

In these notes indices $A, B, \dots$ run from $1,...,n^2$. We freely raise and lower these indices.

We use the following conventions for Yang–Mills transformations of the non-Abelian Yang–Mills multiplet:

$$\delta \theta^A = f^A{}_{BC} \Lambda^B \theta^C, \tag{A1}$$

$$\delta V_i^A = -\mathcal{D}_i \Lambda^A, \tag{A2}$$

$$\mathcal{D}_i \theta^A = \partial_i \theta^A + f^A{}_{BC} V_i^B \theta^C, \tag{A3}$$

$$F_{ij}^A = \partial_i V_j^A - \partial_j V_i^A + f^A{}_{BC} V_i^B V_j^C, \tag{A4}$$

$$\mathcal{D}_{[i} \mathcal{D}_{j]} \theta^A = \tfrac{1}{2} f^A{}_{BC} F_{ij}^B \theta^C. \tag{A5}$$

The U(n) generators are Hermitian $n \times n$ matrices. Our normalization for the trace of two U(n)-generators is

$$\mathrm{tr}\, T_A T_B = \delta_{AB}. \tag{A6}$$

In general, we write for the product of two U(n) generators,

$$T_A T_B = +(d_{ABC} + i f_{ABC}) T_C, \tag{A7}$$

where d and f are symmetric and antisymmetric in AB, respectively. We recognize that

$$\begin{aligned} [T_A, T_B] &= 2 i f_{ABC} T_C, \\ \{T_A, T_B\} &= 2 d_{ABC} T_C. \end{aligned} \tag{A8}$$

From this we conclude that

$$\begin{aligned} \mathrm{tr}[T_A, T_B] T_C &= 2 i f_{ABC}, \\ \mathrm{tr}\{T_A, T_B\} T_C &= 2 d_{ABC}. \end{aligned} \tag{A9}$$

This tells us that in fact f is completely antisymmetric, and d completely symmetric in ABC.

Then we have the Jacobi identity and its generalizations. These follow from

$$\begin{aligned} &[[T_A, T_B], T_C] + [[T_B, T_C], T_A] + [[T_C, T_A], T_B] = 0, \\ &[\{T_A, T_B\}, T_C] + [\{T_B, T_C\}, T_A] + [\{T_C, T_A\}, T_B] = 0, \\ &[T_C, [T_A, T_B]] = \{T_B, \{T_C, T_A\}\} - \{T_A, \{T_B, T_C\}\}. \end{aligned} \tag{A10}$$

From these we derive the following identities for the f and d tensors:

$$f_{ABE} f_{ECD} + f_{BCE} f_{EAD} + f_{CAE} f_{EBD} = 0, \tag{A11}$$

$$d_{ABE} f_{ECD} + d_{BCE} f_{EAD} + d_{CAE} f_{EBD} = 0, \tag{A12}$$

$$f_{ABE} f_{ECD} = d_{CAE} d_{BED} - d_{CBE} d_{AED}. \tag{A13}$$

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